SECTION 10.7: RATIO AND ROOT TESTS

The Limit Comparison Test (LCT) is helpful when a given series resembles a p-series. The tests we discuss in this section measure how closely as series resembles a geometric series.

Recall a geometric series has the form $a + ar + ar^2 + ar^3 + ...$

Geometric series are completely characterized by the first term, a and the common ratio: $r = \dfrac{a_{n+1}}{a_n}$.

Recall a geometric series converges precisely when $|r| < 1$ and diverges otherwise.

Each of the tests in this section checks to see if the terms of a series has a 'common ratio' in a limiting sense.

THE RATIO TEST: Suppose $a_k \neq 0$ and $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L$.

- If $L < 1$, the series $\displaystyle\sum_{k}^{\infty} a_k$ converges absolutely.
- If $L > 1$ or if $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \infty$, the series $\displaystyle\sum_{k}^{\infty} a_k$ diverges.
- If $L = 1$, the series $\displaystyle\sum_{k}^{\infty} a_k$ may converge or may diverge; the ratio test is inconclusive in this case.

EXAMPLE 1: Use the ratio test to analyze the following series.

1. **(VIDEO)** $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}$ — Ans: converges (absolutely)
2. $\displaystyle\sum_{k=1}^{\infty}\frac{\pi^k}{(2k)!}$ — Ans: converges (absolutely)
3. **(VIDEO)** $\displaystyle\sum_{k=1}^{\infty} k!\,e^{-k}$ — Ans: diverges

EXAMPLE 2: Show that the ratio test is inconclusive when applied to the following series:

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ — Ans: $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$
2. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ — Ans: $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$

Another way we can 'get at' the common ratio of a geometric series is to take the nth root of the terms.

An explicit formula for the sequence $a, ar, ar^2, ar^3, \ldots$ is $a_k = a\, r^k,\ k \geq 0$. Provided $a \neq 0$:

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{|ar^k|} = \lim_{k\to\infty} \sqrt[k]{|a|}\,\sqrt[k]{|r^k|} = \lim_{k\to\infty} |a|^{1/k}\left|r^k\right|^{1/k} = \lim_{k\to\infty} |a|^{1/k}\,|r|^{k/k} = 1\cdot|r| = |r|$$

This calculation gives rise to the following:

THE ROOT TEST: Suppose $a_k \neq 0$ and $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = L$.

- If $L < 1$, the series $\sum\limits_{k}^{\infty} a_k$ converges absolutely.

- If $L > 1$ or if $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \infty$, the series $\sum\limits_{k}^{\infty} a_k$ diverges.

- If $L = 1$, the series $\sum\limits_{k}^{\infty} a_k$ may converge or may diverge; the ratio test is inconclusive in this case.

EXAMPLE 3: Use the root test to analyze the following series.

1. $\displaystyle\sum_{k=3}^{\infty}\left(\frac{4-k}{2k+1}\right)^{3k}$ Ans: converges (absolutely)

2. **(VIDEO)** $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^2}$ Ans: diverges

EXAMPLE 4: Show that the root test is inconclusive when applied to the following series:

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ Ans: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = 1$.

2. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ Ans: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = 1$.

HOMEWORK: Section 10.7: 9 - 57 odd.